

Conformal Symmetry for General Black Holes

Mirjam Cvetič^{1,2} and Finn Larsen³

¹Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA-19104, USA.

²Center for Applied Mathematics and Theoretical Physics, University of Maribor, Maribor, Slovenia.

³Michigan Center for Theoretical Physics, 450 Church St., Ann Arbor, MI-48109, USA.

Abstract

We show that the warp factor of a generic asymptotically flat black hole in five dimensions can be adjusted such that a conformal symmetry emerges. The construction preserves all near horizon properties of the black holes, such as the thermodynamic potentials and the entropy. We interpret the geometry with modified asymptotic behavior as the “bare” black hole, with the ambient flat space removed. Our warp factor subtraction generalizes hidden conformal symmetry and applies whether or not rotation is significant. We also find a relation to standard AdS/CFT correspondence by embedding the black holes in six dimensions. The asymptotic conformal symmetry guarantees a dual CFT description of the general rotating black holes.

1. Introduction

Ever since the early days of string theory it has been speculated that a 2D CFT could be responsible for the microscopic interpretation of black hole entropy. The advent of precise realizations of this vision in the context supersymmetric black holes has made the speculations even more appealing, also in settings far from extremality. Indeed, in the string theory community it is widely assumed that the black hole entropy can be accounted for quite generally, in much the same way as it has been for supersymmetric black holes; and this purportedly responds to Hawking’s challenge to quantum mechanics, embodied in the information loss paradigm. However, despite the optimistic conventional wisdom, little concrete progress has been towards a CFT interpretation of black holes far from extremality.

One of the challenges faced by any attempt to make a CFT interpretation precise for general black holes is that generic black holes have negative specific heat. For example, this is the case for most Kerr black holes, including Schwarzschild black holes. This feature of black hole thermodynamics reflects the physical coupling between the internal structure of the black hole and modes that escape to infinity. In order to focus on the black hole “by itself” one must necessarily imagine enclosing the black hole in a box that reflects the emanating radiation and returns it to the black hole, thus creating an equilibrium system. This complication must be taken into account in any precise discussion of black hole thermodynamics, but its necessity is especially imposing if one seeks a dual CFT description, since unitary CFTs always have positive specific heat. In this paper we respond to this necessity in a manner that incorporates several other attractive ideas.

An apparently unrelated clue to the internal structure of black holes involves a massless scalar field probing a general black hole background. It was noticed a long time ago that the wave equation in this setting has remarkable simplifications even for general black holes [1]. In particular there is an $SL(2, \mathbb{R})^2$ symmetry, when certain terms are removed. The offending terms are indeed negligible in many special cases, including the near extreme limit (the AdS/CFT correspondance) [2,1], the near extreme rotating limit (the Kerr/CFT correspondance) [3,4], and the low energy limit [1,5]. However, in general there is no $SL(2, \mathbb{R})^2$ symmetry, just like not all black holes geometries have a near horizon AdS_3 component. This would seem to doom a CFT interpretation of the general case. However, the recent proposal dubbed “hidden conformal symmetry” asserts that the conformal symmetry suggested by the massless wave equation is useful generally after all [5]— it is just

that it is spontaneously broken. This approach has been developed by many researchers, including [6].

The main technical result of this paper is that we construct the geometry corresponding to the wave equation exhibiting $SL(2, \mathbb{R})^2$ symmetry. In other words, we find the geometrical counterpart to the omission of terms violating $SL(2, \mathbb{R})^2$ in the wave equation. We refer to the resulting geometry as the “subtracted geometry”, since it corresponds to removing certain terms in an overall warp factor. The physical interpretation we propose for the subtraction procedure is that it corresponds to enclosure of the black hole in a box: it is the asymptotic Minkowski space that cannot be attributed to the black hole that is being subtracted.

As we have noted, a box delimiting the black hole from its surroundings is inevitable if we seek to identify a dual CFT. The added value offered by the specific box we construct is that it preserves conformal invariance and it is consistent with separation of variables. The subtracted geometry has the same thermodynamic potentials and entropy as the full geometry, and it employs the same time. The only part of the geometry that has been changed is a certain warp factor, and that only in its asymptotic behavior.

The wave equation for a massless scalar field probing the subtracted geometry exhibits $SL(2, \mathbb{R})^2$ symmetry, by construction. However, the geometrical interpretation of this symmetry remains obscure *a priori*. Progress can be made by lifting the geometry with subtracted conformal factor from 5D to 6D. Indeed, the subtracted geometry is recognized after the lift as locally $AdS_3 \times S^3$ geometry. Moreover, the global identifications are such that the AdS_3 factor is precisely the BTZ black hole; and the S^3 is fibered over the AdS_3 in the manner familiar from rotating black holes near extremality [7]. We stress again that, here, we identify these features in the geometry of black holes that are generally far from extremality.

In this paper we focus on the “mesoscopic” analysis of black holes, ie. we seek to infer features of the microscopic theory from the classical geometry. However, the structure that we pursue may persist in the full quantum theory. Given that supersymmetric black holes are described in detail by purely holomorphic CFTs with large central charge we expect that other black holes are similarly described by CFTs with large central charge, albeit no longer holomorphic ones. The semi-classical level matching condition that applies to the black hole entropy all the way off extremality is encouraging for this program [8], as is the (related) semiclassical quantization condition on the areas of the black hole horizons [1,9].

This paper is organized as follows. In section 2 we review the general 5D black hole in string theory, with three charges and two angular momenta. Specifically, we need the representation of the metric as a 4D base with time appearing as a $U(1)$ fibration. We derive the thermodynamic potentials for a large family of geometries taking this form. In section 3 we present the subtracted metric and the wave equation in this background. These are the key technical results. In section 4, we rewrite the subtracted 5D metric in a 6D form, by introducing an auxiliary coordinate. This gives a linear realization of the conformal symmetry. In section 5 we compare our explicit construction to the hidden conformal symmetry program.

2. General 5D Black Holes in String Theory

In this section we review the canonical family of asymptotically flat string theory black holes in $D = 5$ spacetime dimensions [10]. These black holes are the most general solutions in $N = 4, 8$ string theory, up to duality transformations on the matter sector [11]. We employ the recently uncovered form of the metric as a 4D base with time represented as a $U(1)$ fibration [12].

We also derive thermodynamic potentials and other parameters directly from the geometry. This computation will be organized in order that it guide the subsequent construction of a suitable “box” for these black holes.

2.1. The Geometry

The independent parameters of the black hole are the mass, two angular momenta, and three charges. They are parametrized as

$$\begin{aligned}\frac{4G_5}{\pi}M &= \frac{1}{2}\mu \sum_{i=1}^3 \cosh 2\delta_i , \\ \frac{4G_5}{\pi}Q_i &= \frac{1}{2}\mu \sinh 2\delta_i , \quad (i = 1, 2, 3) , \\ \frac{4G_5}{\pi}J_{R,L} &= \frac{1}{2}\mu(b \pm a) (\Pi_c \mp \Pi_s) ,\end{aligned}\tag{2.1}$$

where

$$\Pi_c \equiv \prod_{i=1}^3 \cosh \delta_i , \quad \Pi_s \equiv \prod_{i=1}^3 \sinh \delta_i .\tag{2.2}$$

We write the 5D metric as a fibration over a 4D base space [12]

$$\begin{aligned} ds_5^2 &= -\Delta_0^{-2/3} G(dt + \mathcal{A})^2 + \Delta_0^{1/3} ds_4^2 , \\ ds_4^2 &= \frac{dx^2}{4X} + \frac{dy^2}{4Y} + \frac{U}{G} (d\chi - \frac{Z}{U} d\sigma)^2 + \frac{XY}{U} d\sigma^2 , \end{aligned} \quad (2.3)$$

where for the black holes we consider

$$\begin{aligned} X &= (x + a^2)(x + b^2) - \mu x , \\ Y &= -(a^2 - y)(b^2 - y) , \\ \Delta_0 &= (x + y)^3 H_1 H_2 H_3 , \\ H_i &= 1 + \frac{\mu \sinh^2 \delta_i}{x + y} , \quad (i = 1, 2, 3) , \\ G &= (x + y)(x + y - \mu) , \\ \mathcal{A} &= \frac{\mu \Pi_c}{x + y - \mu} [(a^2 + b^2 - y)d\sigma - abd\chi] - \frac{\mu \Pi_s}{x + y} (abd\sigma - yd\chi) , \\ U &= yX - xY , \\ Z &= ab(X + Y) . \end{aligned} \quad (2.4)$$

The base space coordinates (x, y, σ, χ) are related to the more familiar radial coordinates as

$$\begin{aligned} x &= r^2 , \\ y &= a^2 \cos^2 \theta + b^2 \sin^2 \theta , \\ \sigma &= \frac{1}{a^2 - b^2} (a\phi - b\psi) , \\ \chi &= \frac{1}{a^2 - b^2} (b\phi - a\psi) . \end{aligned} \quad (2.5)$$

The (r, θ, ϕ, ψ) coordinates are such that the base metric asymptotically approaches flat space in the conventional form

$$\begin{aligned} ds_4^2 &\sim dr^2 + r^2 d\Omega_3^2 , \\ d\Omega_3^2 &= d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 . \end{aligned} \quad (2.6)$$

We assume $b^2 > a^2$ so that $y \in (a^2, b^2)$ as $\theta \in (0, \pi/2)$.

The advantage of employing the radial coordinate x and the polar coordinate y , rather than the more conventional r, θ , is that with this parametrization the radial and polar coordinates appear in a roughly symmetric manner. The azimuthal coordinates σ, χ parametrize the angular isometries.

2.2. Black Hole Thermodynamics

The geometry (2.3) is expressed in terms of a large number of functions:

- i) The radial function X depends only on the radial coordinate x .
- ii) The azimuthal function Y depends on the azimuthal coordinate y .
- iii) The warp-factors Δ_0 and G depend only on the combination $x + y$.
- iv) The remaining functions U, Z and the one-form \mathcal{A} depend on the coordinates x, y independently.

An instructive way to appreciate the geometry is to work out the thermodynamics of black holes taking the general form (2.3). In other words, we assume the dependences on x, y given above, but we will not employ the particular functions (2.4).

None of the coordinates t, χ, σ appear in the metric function, so they all parametrize isometries. The coordinate t is special because it represents time in the asymptotic space-time. Accordingly, the *static limit* is the surface

$$G = 0 , \tag{2.7}$$

, since this is where $g_{tt} = 0$. As this surface is crossed, trajectories along the coordinate t cease to be time-like. The *ergosphere* is the volume inside this surface (but outside the event horizon.)

In the ergosphere, physical (ie. light-like) trajectories at fixed x, y always have a component along one or both of the azimuthal angles χ, σ . Physically this means they must co-rotate along with the rotating black hole. A light-like combination of t, χ, σ can be found as long as the sub-determinant in the $t - \chi - \sigma$ space

$$\det g(t - \chi - \sigma) = -XY , \tag{2.8}$$

remains negative. Since $Y > 0$ (except at the poles¹), this condition identifies the *event horizon* as the (outer component of the) locus

$$X = 0 . \tag{2.9}$$

On the inner side of this surface $X < 0$, so there all linear combinations of t, χ, σ are spacelike. Then physical trajectories must move radially along x : capture by the black hole has become inevitable.

¹ We keep the polar angle y fixed in most computations. We also do not analyze the poles at $Y = 0$ which are quite singular in the present coordinates.

Outside the event horizon we can identify a notion of time at any given x, y by diagonalizing the metric in the (t, χ, σ) space and take the coordinate with a negative eigenvalue as “time”. The metric (2.3) is in fact already in diagonal form. Near the event horizon (2.9) ,

$$ds_5^2 = \Delta_{0+}^{1/3} \left(\frac{dx^2}{4X} + \frac{XY_+}{U_+} d\sigma^2 \right) + \dots , \quad (2.10)$$

where (we assume) $U < 0$ at the horizon. Taking the time and the azimuthal angles imaginary, there is a conical singularity in (2.10) at the event horizon $X = 0$ unless σ has the imaginary period

$$\beta_\sigma = \frac{2\pi}{\partial_x X} \sqrt{\left| \frac{U}{Y} \right|} \Big|_{x=x_+} . \quad (2.11)$$

The regularity condition was determined while keeping the space transverse to σ fixed. Thus $\tilde{\chi} = \chi - \frac{Z}{U} \Big|_{x=x_+} \sigma$ was fixed, implying the imaginary period

$$\beta_\chi = \frac{Z}{U} \Big|_{x=x_+} \beta_\sigma , \quad (2.12)$$

and $\tilde{t} = t + \mathcal{A}_\chi \chi + \mathcal{A}_\sigma \sigma$ was fixed, giving the inverse Hawking temperature

$$\beta_H = - \frac{\mathcal{A}_\chi Z + \mathcal{A}_\sigma U}{U} \Big|_{x=x_+} \beta_\sigma = \frac{2\pi}{\partial_x X} \frac{\mathcal{A}_\chi Z + \mathcal{A}_\sigma U}{\sqrt{-UY}} \Big|_{x=x_+} . \quad (2.13)$$

The angular velocity of the black hole along the angles σ, χ are simply the ratios $\Omega_\sigma = \beta_\sigma / \beta_H$ and $\Omega_\chi = \beta_\chi / \beta_H$. The rotational velocities along $\phi \pm \psi$ are then obtained from the linear transformations given in (2.5). They are

$$\begin{aligned} \Omega_R &= (\Omega_\sigma - \Omega_\chi)(b+a) = \frac{(\beta_\sigma - \beta_\chi)}{\beta_H} (b+a) = \frac{U-Z}{\mathcal{A}_\chi Z + \mathcal{A}_\sigma U} \Big|_{x=x_+} (b+a) , \\ \Omega_L &= (\Omega_\sigma + \Omega_\chi)(b-a) = \frac{(\beta_\sigma + \beta_\chi)}{\beta_H} (b-a) = \frac{U+Z}{\mathcal{A}_\chi Z + \mathcal{A}_\sigma U} \Big|_{x=x_+} (b-a) . \end{aligned} \quad (2.14)$$

We also need the black hole entropy, extracted from the area of the event horizon:

$$A_+ = \int dy d\sigma d\chi \sqrt{\det g_{\hat{i}\hat{j}}} , \quad (2.15)$$

where $g_{\hat{i}\hat{j}}$ are the metric components of the $\{\hat{i}, \hat{j}\} = \{y, \sigma, \chi\}$ coordinates, evaluated at the outer horizon $x = x_+$. The sub-determinant simplifies significantly for any metric with the structure (2.3):

$$\det g_{\hat{i}\hat{j}} = g_{yy} \det(g_{\sigma\sigma} g_{\chi\chi} - g_{\sigma\chi}^2) = -\frac{1}{4YU} (\mathcal{A}_\chi Z + \mathcal{A}_\sigma U)^2 . \quad (2.16)$$

Importantly it is *independent* of the conformal factor Δ_0 (as well as G).

In summary, in this subsection we have analyzed a general geometry of the form (2.3) without specifying all the functions (2.4) in detail. We found the position of the ergosphere (2.7), the position of the event horizon (2.9), the Hawking temperature (2.13), the rotational velocities (2.14), and the black hole entropy (2.15). The main lesson from this computation is the remark that all these physical properties are independent of the warp factor Δ_0 . Accordingly we interpret the warp factor as a property of the surrounding spacetime, and not of the black hole “itself”.

2.3. Explicit Expressions

The geometry we are interested in is specified by the functions (2.4). In this case the horizons at the roots of (2.9) are conveniently expressed as

$$x_{\pm} = \frac{1}{4}[\sqrt{\mu - (a - b)^2} \pm \sqrt{\mu - (a + b)^2}]^2 . \quad (2.17)$$

We mention in passing that the formulae in the preceding subsection all focussed on the event horizon $x = x_+$. Similar formulae clearly apply at the Cauchy horizon $x = x_-$. We have stressed the significance of the inner horizon in earlier work (including [1,8]) and will not comment further on this point in the present work.

For the explicit geometries specified by (2.4) the inverse Hawking temperature (2.13) becomes

$$\beta_H = \frac{1}{2}(\beta_R + \beta_L) , \quad (2.18)$$

where

$$\begin{aligned} \beta_R &= \frac{2\pi\mu}{\sqrt{\mu - (b + a)^2}} (\Pi_c + \Pi_s) , \\ \beta_L &= \frac{2\pi\mu}{\sqrt{\mu - (b - a)^2}} (\Pi_c - \Pi_s) . \end{aligned} \quad (2.19)$$

The combination

$$\mathcal{A}_\chi Z + \mathcal{A}_\sigma U = \mu Y (\Pi_c x + ab \Pi_s) , \quad (2.20)$$

is useful in the computations.

The rotational velocities (2.14) become

$$\begin{aligned} \beta_H \Omega_R &= \frac{2\pi(b + a)}{\sqrt{\mu - (b + a)^2}} , \\ \beta_H \Omega_L &= \frac{2\pi(b - a)}{\sqrt{\mu - (b - a)^2}} . \end{aligned} \quad (2.21)$$

The general formulae for the potentials (2.13), (2.14) apply at any value of the polar angle y . The thermodynamic potentials should not depend on y and our final expressions (2.18), (2.21) indeed do not. This constitutes a check on our computations.

The black hole entropy computed from (2.15) becomes $S_{\text{BH}} = \frac{A_+}{4G_5} = S_R + S_L$ where (in $G_5 = \frac{\pi}{4}$ units):

$$\begin{aligned} S_L &= \pi\mu\sqrt{\mu - (b-a)^2}(\Pi_c + \Pi_s) = 2\pi\sqrt{\frac{1}{4}\mu^3(\Pi_c + \Pi_s)^2 - J_L^2} , \\ S_R &= \pi\mu\sqrt{\mu - (b+a)^2}(\Pi_c - \Pi_s) = 2\pi\sqrt{\frac{1}{4}\mu^3(\Pi_c - \Pi_s)^2 - J_R^2} . \end{aligned} \tag{2.22}$$

The y dependence of the subdeterminant (2.16) cancelled entirely prior to the integration (2.15).

3. The Near Horizon Geometry

For near extreme black holes, the AdS/CFT correspondence can be derived as a limit that decouples excitations in the near horizon region from the asymptotic geometry. This standard construction does not generalize to black holes that are not near extremality. This obstruction is physical: generally modes localized near the horizon couple to those in the asymptotic region.

In this section we propose an alternative construction that addresses this obstacle for general black holes.

3.1. The Subtracted Geometry

The mechanics of our proposal is to modify the warp factor in the geometry (2.3) as

$$\Delta_0 \rightarrow \Delta , \tag{3.1}$$

while maintaining all other aspects of the geometry. We will refer to the resulting metric as the *subtracted* geometry.

It was shown in section 2 that thermodynamic potentials are independent of the warp factor. We interpret this to mean that the substitution (3.1) leaves the *interior of the black hole unchanged*. The feature that changes is the asymptotic behavior of the geometry far from the black hole. This reflects a change the *environment* of the black hole. We will

choose the specific Δ in the subtracted geometry such that couplings between the black hole and modes far away are suppressed.

In the context of the explicit solution (2.4) the subtraction we propose modifies the warp factor from

$$\Delta_0 = \prod_{i=1}^3 (x + y + \mu \sinh^2 \delta_i) , \quad (3.2)$$

to

$$\Delta = \mu^2 [(x + y)(\Pi_c^2 - \Pi_s^2) + \mu \Pi_s^2] . \quad (3.3)$$

We will motivate this choice below, by imposing boundary conditions and the requirement that the wave equation remains separable in the subtracted geometry .

The 5D geometries (2.3) asymptote flat space for large $x = r^2$. We verify this by estimating $\Delta_0 \sim x^3 = r^6$, $G \sim r^4$, $\mathcal{A} \sim 0$, and $ds_4^2 \sim r^{-2} \mathbb{R}^4$. The subtracted warp factor (3.3) increases less fast, as $\Delta \sim x$. Consequently, the red-shift $g_{tt} \sim r^{8/3}$ of the subtracted geometry rises rapidly for large x . This prevents particles with finite energy near the black hole from escaping to infinity. We therefore interpret the subtracted geometry as the near horizon geometry of the black hole.

The subtracted geometry will generally not satisfy the equations of motion unless we also modify the matter supporting the original geometry. For example, the uncharged rotating solution (the Myers-Perry black hole) no longer satisfies the vacuum Einstein equations after the warp factor is changed from (3.2) to (3.3). In this situation the Einstein equation acting on the subtracted geometry determines the matter needed to support the solution. We interpret such additional matter as the physical matter supporting the “wall” that we have introduced to separate the interior of the black hole from the irrelevant modes far away.

Although we will not need to specify explicitly what matter is needed to support the solution we briefly pursue one approach that determines it, in section 3.4.

3.2. The Wave Equation

It is instructive to probe the geometry by a spectator scalar field satisfying the Klein-Gordon wave equation

$$[\frac{1}{\sqrt{-g_5}} \partial_\mu (\sqrt{-g_5} g_5^{\mu\nu} \partial_\nu) - M^2] \Phi = 0 . \quad (3.4)$$

The inversion of the metric needed to render this differential equation explicit for the solution we consider is quite nontrivial; so we defer the details to an appendix. When the dust has settled, probes of the form

$$\Phi \sim e^{-i\omega t + im_R(\phi+\psi) + im_L(\phi-\psi)} , \quad (3.5)$$

are found to satisfy the equation

$$\begin{aligned} & \left[4\partial_x X \partial_x + \frac{x_+ - x_-}{x - x_+} \left(\frac{\beta_R \omega}{4\pi} - m_R \frac{\beta_H \Omega_R}{2\pi} + \frac{\beta_L \omega}{4\pi} - m_L \frac{\beta_H \Omega_L}{2\pi} \right)^2 \right. \\ & - \frac{x_+ - x_-}{x - x_-} \left(\frac{\beta_R \omega}{4\pi} - m_R \frac{\beta_H \Omega_R}{2\pi} - \frac{\beta_L \omega}{4\pi} + m_L \frac{\beta_H \Omega_L}{2\pi} \right)^2 + \mu \omega^2 (1 + \sum_i \sinh^2 \delta_i) + x \omega^2 \\ & \left. + 4\partial_y Y \partial_y + y \omega^2 + \frac{1}{Y} ((a^2 + b^2 - y) \partial_\chi^2 + y \partial_\sigma^2 + 2ab \partial_\sigma \partial_\chi) + \frac{\Delta - \Delta_0}{G} \omega^2 \right] \Phi = M^2 \Delta^{1/3} \Phi . \end{aligned} \quad (3.6)$$

The thermodynamic potentials $\beta_{L,R,H}, \Omega_{R,L}$ were given in (2.19). The wave equation (3.6) pertains to arbitrary warp factor Δ but we wrote the last term on the LHS with explicit reference to the unsubtracted warp factor Δ_0 so that the asymptotically flat black hole is easy to recover.

Separability requires that the terms group into some that depend on the radial coordinate x only and some that depend on the polar angle y only. The first two lines in (3.6) are compatible with separability, as are most terms in the last line of (3.6). In fact, taking $M^2 = 0$ (for now), the only term that obstructs separability is the last term on the LHS of (3.6). Separability is a striking characteristic of many rotating black hole solutions so we will elevate it to a principle that determines the possible warp factors.

The massless wave equation (3.6) will be separable for any warp factor Δ such that

$$\frac{\Delta - \Delta_0}{G} = f_1(x) + f_2(y) . \quad (3.7)$$

The functions $f_{1,2}$ are arbitrary at this point but they are constrained by boundary conditions:

- i) $f_1(x)$ should have no poles at finite radius, or else the subtraction procedure will have introduced additional horizons.
- ii) $f_1(x)$ rising faster than linear for large x *increases* couplings between the black hole region and asymptotic space.

iii) The linear coefficient in $f_1(x)$ just changes normalization of the asymptotic Minkowski space, except in the special case of slope “-1” where it suppresses couplings.

iv) Constants in $f_{1,2}$ can be absorbed into separation constants.

These considerations motivate taking $f_1(x) = -x + \text{const.}$, since this is the only case that will make escape to infinity more difficult, while preserving separability and analyticity. Taking $f_2(y)$ linear as well (so that Δ is a function of the combination $x + y$) and choosing separation constants conveniently we are lead to take

$$\Delta = \Delta_0 - [x + y + \mu(1 + \sum_i \sinh^2 \delta_i)]G . \quad (3.8)$$

This is precisely the warp-factor Δ announced in (3.3). It is interesting that the warp factor remains unchanged at the static limit (2.7) separating the ergosphere from the asymptotic space.

The terminology referring to a “subtracted” geometry is motivated by (3.8): the warp factor Δ of the subtracted geometry is expressed as the warp factor Δ_0 of the asymptotically flat black hole, less terms that are separable.

With our choice of warp factor the wave equation (3.6) in the subtracted geometry separates into the angular Laplacian on the round three sphere S^3 :

$$\left[4\partial_y Y \partial_y + \frac{1}{Y} ((a^2 + b^2 - y)\partial_\chi^2 + y\partial_\sigma^2 + 2ab\partial_\sigma\partial_\chi) \right] \Phi_\Omega = -j(j+2)\Phi_\Omega , \quad (3.9)$$

and a radial equation

$$\begin{aligned} & \left[4\partial_x X \partial_x + \frac{x_+ - x_-}{x - x_+} \left(\frac{\beta_R \omega}{4\pi} - m_R \frac{\beta_H \Omega_R}{2\pi} + \frac{\beta_L \omega}{4\pi} - m_L \frac{\beta_H \Omega_L}{2\pi} \right)^2 \right. \\ & \left. - \frac{x_+ - x_-}{x - x_-} \left(\frac{\beta_R \omega}{4\pi} - m_R \frac{\beta_H \Omega_R}{2\pi} - \frac{\beta_L \omega}{4\pi} + m_L \frac{\beta_H \Omega_L}{2\pi} \right)^2 \right] \Phi_r = j(j+2)\Phi_r . \end{aligned} \quad (3.10)$$

The separation constant j is just the usual angular momentum quantum number. The azimuthal quantum numbers are bounded as $|2m_R|, |2m_L| \leq j$.

The radial wave equation (3.10) for the subtracted geometry is just the hypergeometric equation, with singular points at the outer and inner black hole horizons and at asymptotic infinity. The well-known relation between the hypergeometric equation and $SL(2, \mathbb{R})$ is the first step towards identifying a Virasoro algebra in the general subtracted geometry.

3.3. Approximation vs. Subtraction

The simplified radial equation (3.10) is usually interpreted as an *approximation* to the full answer, applicable when the linear terms in (3.6) are negligible compared with the remaining terms. This interpretation can be justified in many situations that involve a small parameter, such as:

- i) The dilute gas regime (hierarchy between the charges) [2,1].
- ii) The near-extremal Kerr regime (large angular momentum) [13,7].
- iii) Small probe energy compared to all black hole parameters [1,5].

Our procedure changes the geometry from the outset, by introducing the subtracted warp factor (3.3), and then computes the exact wave equation. Although the result is the same, our approach brings several advantages:

- i) The estimates justifying application of the simplified radial equation (3.10) generally require assumptions about the black hole parameters, such as near extremality of the black hole. Those assumptions we sidestep here, and so our result applies to the general family of black holes.
- ii). The geometry corresponding to the simplified radial equation is exhibited explicitly, rather than on the level of the wave equation. This facilitates a more thorough analysis, such as generalization to probes with spin.

3.4. Moduli Space

The subtracted geometry does not generally satisfy the equations of motion. Our attitude to this situation is that any enclosure of the black hole necessarily must be formed from matter, and the equations of motion then specify what kind of matter is needed.

Despite this philosophy it is instructive to identify suitable matter by judicious exploitation of moduli space. To do so we start with the black hole at a general point in moduli space, where the warp-factor is

$$\Delta_0 = (x + y)^3 \prod_{i=1}^3 \left(h_i + \frac{\mu \sinh^2 \delta_i}{x + y} \right). \quad (3.11)$$

In this case the geometry is a solution with the usual $N = 2$ matter specified in terms of harmonic functions \mathcal{H}_i with constant terms h_i .

At this point we have a family of geometries with specified matter and warp factors parametrized by the values of h_i . The next step is to adjust the moduli such that ($s_i \equiv \sinh \delta_i$)

$$\begin{aligned} h_1 h_2 h_3 &= 0 , \\ h_1 h_2 s_3^2 + h_2 h_3 s_1^2 + h_3 h_1 s_2^2 &= 0 , \\ h_1 s_2^2 s_3^2 + h_2 s_1^2 s_3^2 + h_3 s_1^2 s_2^2 &= \Pi_c^2 - \Pi_s^2 . \end{aligned} \tag{3.12}$$

At this point in moduli space the warp factor (3.11) takes the “subtracted” form (3.3).

For given boost parameters δ_i , the conditions (3.12) constitute three equations for three variables so we expect that they have a solution. Indeed, we can find solutions by breaking the cyclic symmetry between the three charges. One of the solutions is

$$\begin{aligned} h_1 &= h_2 = 0 , \\ h_3 &= \frac{1}{s_1^2 s_2^2} (\Pi_c^2 - \Pi_s^2) . \end{aligned} \tag{3.13}$$

This is evidently a rather singular point in moduli space, as one would expect since it corresponds to a change in asymptotic behavior. Indeed, it can be interpreted as a limit where two physical charges are taken large with the third finite; and this is just the standard decoupling limit that identifies an AdS_3 near horizon geometry in the 5D black hole.

As we have stated repeatedly, the precise matter supporting the boundary conditions are in fact not important to us. The virtue of the implicit construction of suitable matter, by taking a singular limit in moduli space, is that it provides evidence that the required matter will in fact physically sensible. It also guarantees that our general computation will reduce to standard AdS/CFT results (such as [7]) when two charges are large.

4. Linear Realization of Conformal Symmetry

Let us summarize the situation up to this point: the wave equation in the general black hole geometry (2.3) has several nice properties, such as separability; but it does not quite reduce to hypergeometric form. This is remedied in the “subtracted” geometry, where the warp factor has been changed from (3.2) to (3.3).

Now, the hypergeometric wave equation ensures that there are $SL(2, \mathbb{R})^2$ generators acting on scalar probes of the subtracted geometry. We want to understand the nature of this symmetry and, if possible, extend it to a full conformal symmetry. An illuminating way to proceed is to embed the subtracted 5D black hole geometry in an auxiliary 6D geometry.

4.1. The Auxiliary 6D Geometry

The subtracted geometry remains intricate, as one expects for a charged rotating black hole. The main complication is that the radial coordinate x and the polar coordinate y couple extensively. In appendix A we recast the full metric in a form with radial/temporal terms that depend on x only, angular terms that depend on y only, and a single term encoding the coupling between x and y . In appendix B we show how the remaining coupling between x and y can be eliminated as well, by introducing an auxiliary direction parametrized by a coordinate α . The 6D auxiliary geometry resulting from this construction is

$$\begin{aligned}\ell^{-2}ds_6^2 &= \Delta\left(\frac{1}{\mu}d\alpha + \mathcal{B}\right)^2 + \Delta^{-1/3}ds_5^2 \\ &= \Delta\left(\frac{1}{\mu}d\alpha + \mathcal{B}\right)^2 - \Delta^{-1}G(dt + \mathcal{A})^2 + ds_4^2 ,\end{aligned}\tag{4.1}$$

where the KK-field along α is

$$\mathcal{B} = \frac{1}{\Delta} \left[\mu((a^2 + b^2 - y)\Pi_s - ab\Pi_c)d\sigma + \mu(y\Pi_c - ab\Pi_s)d\chi - \frac{\Pi_s\Pi_c}{\Pi_c^2 - \Pi_s^2}dt \right] .\tag{4.2}$$

The arbitrary length scale ℓ^{-2} was introduced on the LHS of (4.1) in order that dimensions work out correctly.

The auxiliary direction is just a formal device (for now): the 5D wave functions are in 1-1 correspondence with 6D wave functions that are independent of α . To see this note that

$$\ell^{-4}\sqrt{-g_6}g_6^{\mu\nu} = \sqrt{-g_5}g_5^{\mu\nu} ,\tag{4.3}$$

for μ, ν in the 5D space. Massless 6D fields independent of α thus satisfy precisely the same wave equation as massless 5D fields.

The 6D representation (4.1) is aesthetically pleasing in that the awkward fractional powers of the warp factor Δ have been removed. A related physical simplification is the trivial overall conformal factor $g_6 = -\ell^{12}$. It implies that *massive* fields coupling minimally to the 6D metric

$$\left[\frac{1}{\sqrt{-g_6}}\partial_I(\sqrt{-g_6}g_6^{IJ}\partial_J) - M_6^2\right]\Phi = 0 ,\tag{4.4}$$

satisfy a separable wave equation of hypergeometric form. This contrasts with massive fields coupling minimally to the 5D metric: the RHS of (3.6) obstructs separation of variables by coupling radial and angular directions.²

² Massive particles in the unsubtracted geometry with *diagonal* charges separate in 5D as well since then Δ_0 is the cube of a linear function; but this case does not reduce to hypergeometric form [12].

In section 2 we determined the physical temperature of the black hole using a regularity condition at the Euclidean horizon. The computation applies to the 6D geometry (4.1) as well. The condition that the line element $\alpha + \mu\mathcal{B}$ be kept fixed as the horizon is circumnavigated then determines the periodicity of α as

$$\begin{aligned}\beta_\alpha &= -\mu(\mathcal{B}_\sigma\beta_\sigma + \mathcal{B}_\chi\beta_\chi + \mathcal{B}_t\beta_H)_{x=x_+} \\ &= \frac{\Pi_s + \frac{ab}{x_+}\Pi_c}{\Pi_c^2 - \Pi_s^2}\beta_\sigma \\ &= \frac{\pi}{(\Pi_c - \Pi_s)\sqrt{\mu - (b+a)^2}} - \frac{\pi}{(\Pi_c + \Pi_s)\sqrt{\mu - (b-a)^2}}.\end{aligned}\tag{4.5}$$

The expression (4.2) for \mathcal{B} depends nontrivially on the polar coordinate y but such dependence cancels in (4.5), as it should for the thermodynamic potential β_α . This gives a non-trivial check on our computations.

4.2. Factorization

As advertized in the beginning of this section, the 6D geometry (4.1) separates variables manifestly. To see this, we simply expand the functions and collect terms (some details are in the appendices). We find

$$\ell^{-2}ds_6^2 = -\frac{X}{\mu^2 S}dt^2 + \frac{dx^2}{4X} + S(d\alpha - \frac{q_t}{S}dt)^2 + \frac{dy^2}{4Y} + \frac{Y}{y}d\tilde{\sigma}^2 + y(d\tilde{\chi} - \frac{ab}{y}d\tilde{\sigma})^2, \tag{4.6}$$

where S is a linearly transformed version of the radial coordinate x

$$S = x(\Pi_c^2 - \Pi_s^2) + 2ab\Pi_c\Pi_s - (a^2 + b^2 - \mu)\Pi_s^2, \tag{4.7}$$

the potential q_t is

$$q_t = -\frac{ab(\Pi_c^2 + \Pi_s^2) - (a^2 + b^2 - \mu)\Pi_s\Pi_c}{\mu(\Pi_c^2 - \Pi_s^2)}, \tag{4.8}$$

and the shifted azimuthal coordinates are,

$$\begin{aligned}d\tilde{\chi} &= d\chi - \frac{\Pi_s}{\mu(\Pi_c^2 - \Pi_s^2)}dt + \Pi_c d\alpha, \\ d\tilde{\sigma} &= d\sigma - \frac{\Pi_c}{\mu(\Pi_c^2 - \Pi_s^2)}dt + \Pi_s d\alpha.\end{aligned}\tag{4.9}$$

Not only does (4.6) separate variables manifestly directly in the geometry: the metric is locally $\text{AdS}_3 \times S^3$ even for the general black holes we study. This is important because

it immediately implies that the $SL(2, \mathbb{R})^2$ isometries are enhanced to Virasoro algebras [14,15]. This is what we wanted to show.

The generators of the AdS_3 Virasoro algebras depend in an essential manner on the spatial isometry parametrized by the coordinate α . In physical terms, the oscillations that the Virasoro algebras act on are right and left moving waves moving along the α directions. The status of such states is uncertain since α was introduced as an auxiliary variable.

Having mentioned this important caveat, we should also recall that once there is an AdS_3 component of the geometry, the entropy is in fact *guaranteed* to work out: modular invariance in the CFT is geometrized directly in the BTZ black hole [16,17], and so the black hole entropy is computed by the low energy thermal modes in a manner that always gives the “right” result. The important point will therefore not be to establish a numerical agreement for the black hole microstates, but rather to understand whether these states are in fact physical.

4.3. BTZ Interpretation

It is instructive to express the first three terms in (4.6) in the standard BTZ form

$$ds_{\text{BTZ}}^2 = -N^2 dt^2 + N^{-2} dR^2 + R^2 (d\phi_{\text{BTZ}} + \frac{4G_3 J_3}{R} dt)^2, \quad (4.10)$$

where

$$N^2 = \frac{(R^2 - R_+^2)(R^2 - R_-^2)}{\ell^2 R^2} = \frac{R^2}{\ell^2} - 8G_3 M_3 + \frac{16G_3^2 J_3^2}{R^2}. \quad (4.11)$$

Comparison with (4.6) identifies the time coordinates and gives simple rescalings for the remaining coordinates

$$S = \frac{R^2 \mu^2 (\Pi_c^2 - \Pi_s^2)^2}{\ell^4}, \quad (4.12)$$

$$\phi_{\text{BTZ}} = \frac{\mu (\Pi_c^2 - \Pi_s^2)}{\ell} \alpha.$$

The horizon loci R_{\pm}^2 give the effective BTZ black hole parameters

$$8G_3 M_3 = \frac{\ell^2}{\mu^2 (\Pi_c^2 - \Pi_s^2)^2} [(\mu - a^2 - b^2)(\Pi_c^2 + \Pi_s^2) + 4ab\Pi_c\Pi_s], \quad (4.13)$$

$$4G_3 J_3 = \frac{\ell^3}{\mu^2 (\Pi_c^2 - \Pi_s^2)^2} [ab(\Pi_c^2 + \Pi_s^2) - (a^2 + b^2 - \mu)\Pi_s\Pi_c].$$

5. Black Hole Microstate Counting

The $SL(2, \mathbb{R})^2$ symmetry of the subtracted geometry is non-abelian so all generators are normalized uniquely. The hidden conformal symmetry approach combines this property

with the known periodicity of the azimuthal angles and infer notions of temperature in the dual CFT description. Alternatively, we can take advantage of the auxiliary coordinate we have introduced to identify an explicit Virasoro algebra using standard AdS/CFT technology. In this section we develop both these approaches.

5.1. Hidden Conformal Symmetry

The non-abelian nature of $SL(2, \mathbb{R})^2$ determines the properly normalized $U(1)$ generators as³

$$\begin{aligned}\mathcal{R}_3 &= \frac{i}{4\pi} (\beta_R \partial_t + \beta_H \Omega_R (\partial_\phi + \partial_\psi)) , \\ \mathcal{L}_3 &= \frac{i}{4\pi} (\beta_L \partial_t + \beta_H \Omega_L (\partial_\phi - \partial_\psi)) .\end{aligned}\tag{5.1}$$

We may realize these operators as $\mathcal{R}_3 = i\partial_{t_R}$, $\mathcal{L}_3 = i\partial_{t_L}$ by introducing the coordinates

$$\begin{aligned}t_R &= \frac{4\pi}{\beta_R} \left(t - \frac{\beta_L}{2\beta_H \Omega_L} (\phi - \psi) \right) , \\ t_L &= \frac{4\pi}{\beta_L} \left(t - \frac{\beta_R}{2\beta_H \Omega_R} (\phi + \psi) \right) .\end{aligned}\tag{5.2}$$

The “hidden” part of hidden conformal symmetry refers to the observation that such coordinates are globally ill-defined, because of periodicity along the azimuthal angles. Comparing the periodicities of t_R, t_L obtained this way with the standard CFT definition of temperature $z \equiv z + 4\pi^2 i T^{\text{CFT}}$ we find

$$\begin{aligned}T_R^{\text{CFT}} &= \frac{1}{\beta_R} \frac{\beta_L}{\beta_H \Omega_L} , \\ T_L^{\text{CFT}} &= \frac{1}{\beta_L} \frac{\beta_R}{\beta_H \Omega_R} .\end{aligned}\tag{5.3}$$

Note that these dimensionless CFT temperatures differ from the dimensionfull physical temperatures $T_{R,L}^{\text{phys}} = \beta_{R,L}^{-1}$ that govern Hawking radiation.

If a CFT with the temperatures (5.3) is responsible for the black hole entropy (2.22) it must be that

$$S_R = \frac{\pi^2}{3} c_R T_R^{\text{CFT}} = \pi \mu \sqrt{\mu - (b+a)^2} (\Pi_c - \Pi_s) .\tag{5.4}$$

³ The properly normalized generators were introduced already in [1]. Eqs. 56 and 57 of [1] are $\mathcal{R}_3, \mathcal{L}_3$, and the remaining generators are given in the preceding equations.

Simplifying this expression using the parametric expressions for potentials (2.19),(2.21) and the physical black hole parameters (2.8) we find the central charge

$$c_R = 12|J_L| . \quad (5.5)$$

Similarly, S_L works out correctly if $c_L = 12|J_R|$.

The apparent interchange up of R and L is surprising but it could be correct. For example, in the extremal limit where $S_L \rightarrow 0$ with no charges the full entropy $S_R = 2\pi\sqrt{J_L^2 - J_R^2}$ which is well described by a CFT with $c_R = 12|J_L|$ and R-charge identified with J_R . Likewise it is also acceptable *a priori* that the right and central charges are different such that the underlying theory is chiral. This could well be a feature required for a CFT describing black holes with angular momenta $J_R \neq J_L$ and arbitrary charges.

An important challenge to the hidden conformal symmetry program is the non-uniqueness of the coordinates (5.2) realizing the conformal generators. The specific realization (5.2) is such that symmetry between R and L is preserved but it would be equally acceptable to take

$$t'_L = \frac{2\pi}{\beta_H \Omega_L} (\phi - \psi) , \quad (5.6)$$

while keeping the t_R in (5.2). In this basis the temperatures are

$$T'^{\text{CFT}}_L = \frac{1}{\beta_H \Omega_L} , \quad (5.7)$$

with T^{CFT}_R still given by (5.3). Assuming again that the entropies (2.22) are reproduced in the CFT, the central charges are inferred as $c_R = c_L = 12|J_L|$. This basis thus suggests a non-chiral CFT.

It is clear that many bases realize the conformal generators and that their periodicity conditions motivate a range of central charges. The primed basis above is natural in the near extreme limit of a rotating black hole with all rotation along the J_L direction but it is clearly an awkward choice for black holes rotating mostly along the J_R direction. This ambiguity challenges the proposal that a CFT with these central charge assignments might account for the entropy of all black holes.

5.2. The Long String Picture

Our embedding of the black hole geometry into 6D suggests another avenue for understanding the black hole entropy. In this approach the excitations responsible for the entropy are along the auxiliary coordinate α rather than the azimuthal angles in physical space. To develop this scenario we must specify the AdS_3 radius ℓ which is not determined by the wave equation and also compute the effective 3D coupling G_3 which we have kept arbitrary for now.

We first rewrite the answer we seek as follows. We describe R and L entropies (2.22) in terms of dilute gasses with physical temperatures (2.19) so

$$S_{R,L} = \frac{\pi^2}{3} cT_{R,L}^{\text{phys}} \mathcal{R} , \quad (5.8)$$

where for both R and L chiralities

$$c\mathcal{R} = 6\mu^2(\Pi_c^2 - \Pi_s^2) . \quad (5.9)$$

The length scale $2\pi\mathcal{R}$ is the volume of the 1D gas, needed to transform from physical temperatures to the dimensionless CFT temperature. A microscopic understanding amounts to accounting for (5.9).

The effective 3D coupling constant G_3 is identified by comparing the 6D \rightarrow 3D reduction on S^3 with the 6D \rightarrow 5D reduction on the circle parametrized by α [18]. Assigning again the length scale $2\pi\mathcal{R}$ to the effective string direction we find

$$G_3 = G_5 \frac{2\pi\mathcal{R}}{V(S^3)} = \frac{4G_5}{\pi} \cdot \frac{\mathcal{R}}{4\ell^3} . \quad (5.10)$$

The Brown-Henneaux formula then gives

$$c\mathcal{R} = \frac{3\ell}{2G_3} \cdot \mathcal{R} = 6\ell^4 , \quad (5.11)$$

in string theory conventions (see [19]) where $G_5 = \frac{\pi}{4}$. Combining formulae, (5.9) implies the AdS_3 scale

$$\ell^4 = \mu^2(\Pi_c^2 - \Pi_s^2) . \quad (5.12)$$

The length scale \mathcal{R} drops out, as it should due to conformal invariance in the boundary theory. The invariant scale that we need to explain is the AdS_3 radius (5.12).

The key ingredient we have at our disposal is the result from section 2.2 that the thermodynamic potentials and the black hole entropy are independent of the warp factor and so on the subtraction procedure. Since we have assigned periodicity $2\pi\mathcal{R}$ to the periodic length scale in the CFT, the BTZ angle ϕ_{BTZ} has periodicity $2\pi\mathcal{R}/\ell$; and so the BTZ entropy becomes

$$\begin{aligned} S_{\text{BTZ}} &= \frac{A_3}{4G_3} = \frac{2\pi R_+}{4G_3} \cdot \frac{\mathcal{R}}{\ell} \\ &= 2\pi \cdot \frac{\ell^4}{\mu(\Pi_c^2 - \Pi_s^2)} \cdot \left(\sqrt{\mu - (a-b)^2}(\Pi_c + \Pi_s) + \sqrt{\mu - (a+b)^2}(\Pi_c - \Pi_s) \right) . \end{aligned} \quad (5.13)$$

Consistency with the 5D black hole entropy (2.22) then implies (5.12). This inference constitutes a *derivation* of the AdS_3 length scale which in turn accounts for the black hole entropy in full generality.

At this point we can apply standard AdS/CFT technology to derive further results. For example, the conformal weights assigned to the black hole are $h_{L,R} = \frac{M_3\ell \pm J_3}{2}$ where the BTZ parameters are (4.13).

A particularly appealing feature of the AdS_3 description is the manifest modular invariance, realized geometrically as interchange of the (Euclidean) t -direction and the auxiliary α -direction [16]. The symmetry between these directions is apparent already in the thermodynamic potential (4.5) which we can write as

$$\beta_\alpha = \frac{1}{2\mu(\Pi_c^2 - \Pi_s^2)}(\beta_R - \beta_L) . \quad (5.14)$$

The relation (4.12) to ϕ_{BTZ} which has known periodicity $2\pi\mathcal{R}/\ell$ then determines

$$\mathcal{R} = \frac{\beta_R - \beta_L}{4\pi} . \quad (5.15)$$

This quantity is interpreted physically as a chemical potential.

The S^3 is fibered over the AdS_3 base space, as encoded in the shifted angles (4.9). We interpret this as a generalization of the corresponding effect for near extreme black holes [7,20,21,22]: the effective string in the (t, α) plane has been boosted by $\cosh \delta_0 = \Pi_c / \sqrt{\Pi_c^2 - \Pi_s^2}$, and the size of the sphere is set by the scale ℓ in (5.12)⁴. These shifts encode the energy cost of carrying angular momentum.

⁴ The shifts (4.9) generalize eqs 13 and 14 in [7]. They reduce precisely to those in the limit where $\delta_{1,2} \gg 1$.

For near extreme black holes with large charges there are different limits where the description becomes effectively free. In the case of the extreme D1-D5 system the asymptotic behavior in one limit is controlled by energy $h_{\text{eff}} = Q_1 Q_5 p$, with each string contributing of order one to the level; but in another limit the energy $h_{\text{eff}} = p$, with each string contributing fractions of order $1/Q_1 Q_5$ [23]. In the near extreme limit the scale (5.12) increases rapidly with boost parameter and reduces precisely to the “long” string scale, the one that gives rise to maximal fractionation. The description in this subsection is thus a generalization of the long string picture.

The hidden conformal symmetry approach interprets the symmetries differently from the long string picture.⁵ In the non-chiral version of hidden conformal symmetry the division of the entropy into R and L parts agree, as do the corresponding assignments of temperatures. The remaining physical difference is then the length scale that relates the physical temperature to the CFT temperature. Those are genuinely different.

6. Discussion

We conclude with a short discussion of our main results and some future directions that they open:

i) *Subtracted Warp Factor*: we modify the warp factor while maintaining all the remaining parts of the metric. It would be interesting to generalize this construction to other contexts.

We accepted that the subtracted metric does not satisfy the equations of motion, arguing that this is a physical property of adding an enclosure that decouples the black hole from the asymptotically flat space. It would be nice to understand the implied supporting matter in more detail. Alternatively, in some cases one may prefer to argue that the subtracted warp factor is a good approximation in the important region.

ii) *The 6D Lift*: we lifted the subtracted geometry to one dimension higher, by adding an auxiliary coordinate α . This simplified the otherwise very complicated geometry enormously. We anticipate that this type of lift to higher dimensions will be useful in many contexts. For example, it may shed light on the mysterious symmetries enjoyed

⁵ A technical difference that obscures comparisons is that hidden conformal symmetry is developed in the dual modular frame where temperature is identified as imaginary periodicity $4\pi^2 T$, while in the long string picture we identify the temperature with the imaginary periodicity β .

by the 4D Kerr black hole. The auxiliary coordinate also appears to create novel symmetry transformations that mix it with the physical coordinates.

iii) *Black Hole Entropy*: our construction provides a fairly systematic way to identify a conformal symmetry for a very large class of black holes. It should be possible to exploit this result to get a convincing account of the general black hole entropy. We developed two approaches: hidden conformal symmetry (adapting [5] to the present setting) and what we refer to as the long string picture (detailing our earlier work, including [1,8]). In the latter approach we were able to find a quantitative match for the entropy with no free parameters.

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Appendix A. Derivation of the Scalar Wave Equation

We are putting great emphasis on the form of the scalar wave equation. The analogous simplifications are less evident directly in the geometry (2.3). In this appendix we present the steps we take to determine the scalar wave equation from the subtracted geometry.

The poles in the metric (and in the wave equation) provide significant guidance for the explicit manipulations. For example, the base metric ds_4^2 in (2.4) appears to have a pole where $U = 0$ but, upon expansion of the terms, contributions to this pole in fact cancel. We can recast the base metric as

$$ds_4^2 = \frac{1}{G} [U d\chi^2 + ((X + Y)(a^2 + b^2) - U - \mu Y) d\sigma^2 - 2ab(X + Y) d\sigma d\chi] . \quad (\text{A.1})$$

The apparent pole where $G = 0$ similarly cancels in the full 5D metric. To see this, introduce the coordinates

$$\begin{aligned} S &= x(\Pi_c^2 - \Pi_s^2) + 2ab\Pi_c\Pi_s - (a^2 + b^2 - \mu)\Pi_s^2 , \\ T &= y(\Pi_c^2 - \Pi_s^2) - 2ab\Pi_c\Pi_s + (a^2 + b^2)\Pi_s^2 . \end{aligned} \quad (\text{A.2})$$

The coordinates S, T are just linear transformations on the radial and polar coordinates x, y (to which they reduce in the absence of charges). They pick out the radial and polar part of the conformal factor

$$\Delta = \mu^2(S + T) = (\Pi_c^2 - \Pi_s^2)(x + y) + \mu\Pi_s^2 . \quad (\text{A.3})$$

After a straightforward (and only moderately tedious) computation we find

$$\begin{aligned} \Delta^{-1/3} ds_5^2 &= -\frac{G}{\mu^2(S + T)}(dt + \mathcal{A})^2 + ds_4^2 \\ &= -\frac{G}{\mu^2(S + T)}(dt^2 + 2\mathcal{A}dt) + [yd\chi^2 + (a^2 + b^2 - y)d\sigma^2 - 2abd\sigma d\chi + \frac{dy^2}{4Y}] \\ &\quad + \frac{dx^2}{4X} - \frac{1}{S + T}[(\Pi_c y - ab\Pi_s)d\chi - (\Pi_c ab - \Pi_s(a^2 + b^2 - y))d\sigma]^2 . \end{aligned} \quad (\text{A.4})$$

The apparent pole at $G = 0$ cancelled as claimed. The angular terms take a nice form in this equation: the first square bracket is just the round sphere S^3 , and the second one then represents a deformation due to rotation and charges.

To find the wave equation we need to invert the metric and for this purpose (A.4) is not optimal. A good alternate form (which takes some effort to reach) is

$$\Delta^{-1/3} ds_5^2 = \frac{dx^2}{4X} - \frac{X}{\mu^2 S} dt^2 + \frac{dy^2}{4Y} + \frac{Y}{T}(\Pi_s d\chi - \Pi_c d\sigma + \frac{1}{\mu} dt)^2 + \frac{ST}{S + T} \left(\frac{p}{T} - \frac{q}{S} \right)^2 , \quad (\text{A.5})$$

where

$$\begin{aligned} p &= ((a^2 + b^2 - y)\Pi_s - ab\Pi_c)d\sigma + (y\Pi_c - ab\Pi_s)d\chi + \frac{ab(\Pi_c^2 + \Pi_s^2) - (a^2 + b^2)\Pi_s\Pi_c}{\mu(\Pi_c^2 - \Pi_s^2)} dt , \\ q &= -\frac{ab(\Pi_c^2 + \Pi_s^2) - (a^2 + b^2 - \mu)\Pi_s\Pi_c}{\mu(\Pi_c^2 - \Pi_s^2)} dt . \end{aligned} \quad (\text{A.6})$$

This form of the metric is a sum of five complete squares so it is explicitly diagonalized. The inverse metric is therefore just the inverse of each eigenvalue, written in the dual basis:

$$\begin{aligned} \Delta^{1/3} g^{\mu\nu} \partial_\mu \partial_\nu &= 4X \partial_x^2 - \frac{S}{X} \left(\mu \partial_t - \frac{1}{\Pi_s} \partial_\chi + \frac{1}{S} (x\Pi_c + ab\Pi_s) (\partial_\sigma + \frac{\Pi_c}{\Pi_s} \partial_\chi) \right)^2 + 4Y \partial_y^2 \\ &\quad + \frac{T}{Y} \left[\frac{1}{\Pi_s} \partial_\chi - \frac{y\Pi_c - ab\Pi_s}{T} (\partial_\sigma + \frac{\Pi_c}{\Pi_s} \partial_\chi) \right]^2 + \left(\frac{1}{S} + \frac{1}{T} \right) (\Pi_s \partial_\sigma + \Pi_c \partial_\chi)^2 . \end{aligned} \quad (\text{A.7})$$

The determination of the dual basis uses the identity

$$\frac{y}{T} - \frac{x}{S} = \frac{\Pi_s}{\Pi_c^2 - \Pi_s^2} \left(\frac{2ab\Pi_c - (a^2 + b^2)\Pi_s}{T} + \frac{2ab\Pi_c - (a^2 + b^2 - \mu)\Pi_s}{S} \right). \quad (\text{A.8})$$

The determinant of the metric is just $\det g_5 = -\frac{1}{16}\Delta^{2/3}$ so the Laplacian operator becomes

$$\begin{aligned} \partial_\mu(\sqrt{-g_5}g^{\mu\nu}\partial_\nu) &= \partial_x(X\partial_x) - \frac{S}{4X} \left(\mu\partial_t + \frac{\Pi_s(x + a^2 + b^2 - \mu) - ab\Pi_c}{S}\partial_\chi + \frac{x\Pi_c + ab\Pi_s}{S}\partial_\sigma \right)^2 \\ &+ \partial_y(Y\partial_y) + \frac{1}{4YT} [((a^2 + b^2 - y)\Pi_s - ab\Pi_c)\partial_\chi - (y\Pi_c - ab\Pi_s)\partial_\sigma]^2 + \frac{1}{4} \left(\frac{1}{S} + \frac{1}{T} \right) (\Pi_s\partial_\sigma + \Pi_c\partial_\chi)^2 \\ &= \partial_x(X\partial_x) - \frac{1}{4X} [S\mu^2\partial_t^2 + 2\mu(\Pi_s(x + a^2 + b^2 - \mu) - ab\Pi_c)\partial_\chi\partial_t + 2\mu(x\Pi_c + ab\Pi_s)\partial_\sigma\partial_t \\ &+ x\partial_\sigma^2 - (x + a^2 + b^2 - \mu)\partial_\chi^2 - 2ab\partial_\chi\partial_\sigma] + \partial_y(Y\partial_y) + \frac{1}{4Y} [(a^2 + b^2 - y)\partial_\chi^2 + y\partial_\sigma^2 + 2ab\partial_\sigma\partial_\chi]. \end{aligned} \quad (\text{A.9})$$

In the first expression we rewrote the terms so that the limit of vanishing charge $\Pi_s \rightarrow 0$ is manifestly regular. In the second expression we collected term to make is manifest that there are no poles in the wave equation at $S = 0$ and T .

The final expression (A.9) has several notable features:

- i) *Separation of variables*: the radial and angular variables do not couple.
- ii) *Singularity structure*: as we have emphasized, most presentations of the metric and/or the wave equation has a number of spurious singularities (eg. at $U = 0$, $G = 0$, $S = 0$, $T = 0$) but these are all absent in (A.9).
- iii) *Only simple poles*: X given in (2.4) is a quadratic function with two distinct roots (for nonextremal black holes). Thus the poles at $X = 0$ can be decomposed such that the only singularities are simple poles.
- iv) *Spherical symmetry*: the angular Laplacian (the square bracket in the last line) is the same as in flat space. Thus rotation of the black hole has not broken rotational symmetry in the wave equation of the subtracted metric.
- v) *Locality*: the expression in the first square bracket is *linear* in x . This is the feature that suppresses coupling of modes to the asymptotic space. Linearity means the term can be decomposed as exactly two pole terms. The complicated dependence on parameters in these terms is just due to the intricate thermodynamics of these black holes, as expressed succinctly in (2.19),(2.21).

vi) *Hypergeometric Structure*: the kinetic terms have poles only as $X = 0$ (respectively $Y = 0$), the same positions as the simple poles in the potential. This gives the equation its hypergeometric character.

The wave equation for the full metric, without subtraction in the conformal factor, was presented in (3.6)(following [1]). Simplifying properties i), ii), iii) remain; but there are additional terms that obstruct properties iv), v), vi).

The near horizon limit of near extreme Kerr is such that the wave equation enjoys all the simplifying properties above, except spherical symmetry iv) [13],[4]. Several other near horizon limits that may be applied to the wave equation have the effect of restoring all the simplifying properties [1].

Appendix B. Derivation of the 6D Lift

As we have emphasized, the scalar wave equation is separable into radial and angular equations. We want to exhibit this factorization explicitly in the geometry.

The form (A.5) of the geometry is a good starting point. The first two terms depend just on the radial/temporal variables x , t , q , S . The next two terms depend just on the angular variables y , χ , σ , p , T , up to a shift of χ , σ that is linear in t . This latter complication indicates that the angular space is fibered over the radial/temporal one, a feature that is well-known from the description of near extreme rotating black holes [7].

It is thus the last term in (A.5) that encodes the coupling between the radial and angular parts of the geometry. We can simplify this coupling greatly by lifting to the 6D geometry

$$\ell^{-2}ds_6^2 = \Delta\left(\frac{1}{\mu}d\alpha + \mathcal{B}\right)^2 + \Delta^{-1/3}ds_5^2, \quad (\text{B.1})$$

where

$$\mathcal{B} = \frac{\mu}{\Delta}(p + q). \quad (\text{B.2})$$

The 6D KK gauge field (B.2) was designed to factorize the offending term in the geometry, which now takes the form

$$\begin{aligned} \ell^{-2}ds_6^2 = & \frac{dx^2}{4X} - \frac{X}{\mu^2 S}dt^2 + S(d\alpha - \frac{1}{S}q)^2 \\ & + \frac{dy^2}{4Y} + \frac{Y}{T}(\Pi_c d\sigma - \frac{1}{\mu}dt - \Pi_s d\chi)^2 + T(d\alpha + \frac{1}{T}p)^2. \end{aligned} \quad (\text{B.3})$$

This is what we want: the first line depends just on the radial/temporal variables x, t, α, q, S . The second line depends just on the angular variables y, χ, σ, p, T , up to a shift of χ, σ that is linear in t, α . In this representation the angular space is *only* coupled to the radial one by the non-trivial fibration.

The angular terms can be simplified so that the full metric becomes

$$\ell^{-2}ds_6^2 = \frac{dx^2}{4X} - \frac{X}{\mu^2 S}dt^2 + S(d\alpha - \frac{1}{S}q)^2 + \frac{dy^2}{4Y} + \frac{Y}{y}d\tilde{\sigma}^2 + y(d\tilde{\chi} - \frac{ab}{y}d\tilde{\sigma})^2, \quad (\text{B.4})$$

where the shifted coordinates

$$\begin{aligned} d\tilde{\chi} &= d\chi - \frac{\Pi_s}{\mu(\Pi_c^2 - \Pi_s^2)}dt + \Pi_c d\alpha, \\ d\tilde{\sigma} &= d\sigma - \frac{\Pi_c}{\mu(\Pi_c^2 - \Pi_s^2)}dt + \Pi_s d\alpha. \end{aligned} \quad (\text{B.5})$$

In this form the metric is recognized as $\text{BTZ} \times S^3$. Thus our construction associates a locally $\text{AdS}_3 \times S^3$ geometry to the general black holes we study.

A conservative interpretation of the auxiliary coordinate α is that it represents a redundancy that we can introduce without loss of generality. Certainly wave functions that are independent of α satisfy the same wave equation in the 6D geometry (B.3) and in the 5D geometry (A.5). From this point of view we have introduced a redundancy in the description, in order to furnish a linear realization of the symmetries.

A more ambitious (and speculative) interpretation of the coordinate α is to identify it with the physical coordinate along an effective string. In this view the microscopic interpretation of black holes involves dependence on α in an essential manner, with the macroscopic (thermodynamic) description sensitive only to features that are independent of α .

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